THE PSYCHOLOGY OF ERRORS IN ALGEBRA

By PERCIVAL M. SYMONDS
Institute of Educational Research
Teachers College, Columbia University

Ask a teacher why one of his pupils is doing poorly in algebra and you will receive one of three answers: that the pupil is stupid, or that he is careless, or that he is lazy. Whichever of these answers you receive is probably correct, but not very helpful. A teacher who thinks of his poorer pupils as stupid, careless, or lazy does not hold the key for remediation. If the boy is stupid, no remedy can cure him. Better teaching may make a little improvement in his accomplishment, increased motivation may lead to slightly greater achievement, but the net result will be meagre and unsatisfactory. Success in algebra does not come to the stupid boy—we find this evidenced in the high correlation which we find between algebraic ability and intelligence.

Likewise the teacher who diagnoses a cause of failure merely as carelessness, or as laziness, is not likely to do much in preventing such failure. The boy who fails in algebra does so because he does not do the examples. The boy who fails, fails because when confronted with an examination paper he has not become able to manipulate the problems so as to come out with the "correct answer" or even to reach any answer. Somewhere along the line he goes astray, he does the wrong thing—or is unable to do anything. In short he makes errors. Rugg and Clark in *Scientific Method in the Reconstruction of Ninth Grade Mathematics* (p. 81) divide errors into two groups: (1) accidental errors, and (2) recurring errors. By accidental errors these authors refer to errors of reading, writing, following directions, arithmetic, etc. Speaking precisely, these are no more accidental than any other kind of error, and after sufficient search, a satisfactory explanation can almost always be given in the individual case. "An accidental error is probably due to intermittent concentration, to lapses of attention, to instances of mind-wandering . . . one of the fundamental causes of failure in any school study is mental inertia and its accompanying weakness—mind-wandering, day-dreaming." The remedy for this is a more spirited class hour with little opportunity for this scattered at-

tention. The common practice of having examples already worked by the class put on the board and read aloud for the benefit of a minority who need to see the steps undoubtedly is a factor in building this habit of mind-wandering.

But more important for our present consideration are the recurring errors. By these the teacher means errors that a boy consistently makes. They are definitely formed habits, but wrong habits. Rugg and Clark have listed 44 such errors as they have found them to occur in their standardized tests. It is our purpose to show the psychology of these errors, and from this to point out methods of eradicating them, or better, of preventing the formation of these wrong habits.

Thorndike enumerates five characteristics of behavior. The first is multiple response to the same external situation. This is a grimly humorous truth in algebra. In how many ways may a boy respond to the situation $(x^3)^4$! He may write as an answer x^7, or x, or $x^{\frac{4}{3}}$, or x^{81}, or x^{12}. Or consider the problem, "If one part of x is a, what is the other part?" In putting this problem to a boy I received the following successive replies: "one," "it couldn't be," "ax," "xa." The teaching of algebra is made possible by the fact of multiple response on the part of the pupil. The problem of teaching algebra is: How can we ensure for ourself the one response that we desire?

A second characteristic of behavior is that the pupil has a set or determination in being presented with a problem—to get *the* answer. And when he has obtained *an* answer that in any way satisfies this set or determination, he is satisfied that he has *the* answer. There are three common satisfactions that determine in this way a pupil's choice of his answer. One is the social approval evidenced by the teacher or members of the class. I have found in watching boys work that for some the sole criterion for correctness is the nod or expression of the teacher. A favorable nod from the teacher is sufficient to convince this type of boy that he has not only found an answer that satisfies the teacher, but has found the correct answer, even though the answer really is wrong. A negative shake of the head will send the boy back again over his work to find his error. A perfectly neutral attitude on the part of the teacher will make the boy extremely uncomfortable, for it withholds from him his

only check on his result. Such a passive attitude will often evoke a question as, "Did I do it right?" Now this reliance on approval is not wholly bad. Most of our language habits are formed by approval of the correct form and disapproval of the incorrect form. It is only that our algebra is a more individual thing than language, and we expect, indeed, for success we must have, some other means by which the pupil may determine the correct answer independently.

A second "satisfaction" that determines a pupil's choice of his answer is an answer with which he may compare his—the answer book or key. The argument against keys is that the pupil will misuse them and will obtain the result in unfair ways. That is, we expect the pupil not only to get the answer, but to do the work of getting it, and usually in a prescribed way. But this objection is valid only where the work is defined as doing such and such exercises. Give a boy an assignment, "Pass in examples 1-12 on page 86 for tomorrow," and there is every incentive to copy or use answer books wrongly, but give a boy an assignment such as, "Practice these until you can get 12 right in 6 minutes," and the motive for improper use of the answer book is gone.

But somehow we expect our mathematicians to be more independent than this in determining correctness of answers. So a third and more intrinsic satisfaction is that of checking the answer. We all know these checks—after subtracting, adding the difference and bottom number to make the top number, multiplying the divisor and quotient to make the dividend, multiplying the factors to make the number to be factored, substituting the value of the unknown in the equation, substituting any number for the letters in a long and involved identity, etc. Almost all algebraic processes are either thus reversible, or may be checked by substituting. We sometimes think that checking is a troublesome and burdensome "extra" process which takes away time from the real drill at hand, but substituting the value of x in a literal equation often gives as difficult a process as the original solution. We should bear in mind that we *must* give the pupil some way of determining the correctness of his answer; if not teacher approval or keys, then we must allow the pupil the confidence that comes from checking.

Finally, with experience comes a divination of the fitness of an answer—as to the degree or dimensions of the answer, we have the feel as to whether the answer is .01 or 100.

Before leaving this topic we should consider one potent method a boy has of testing the correctness of his answer—namely, whether it comes out even or not. Early in his course, if it has not already been acquired in arithmetic, he becomes suspicious of any answer that does not come out even. In connection with the algebra inquiry, Mr. Orleans has made counts in nine texts in present use, to determine the number of answers that are integers, fractions, decimals, surds, or imaginaries. Following is a tabulation of his results expressed in percentages:

	A	B	C	D	E	F	G	H	I	Total
Integer	62.6	63.7	49.4	61.1	43.0	71.7	59.1	58.1	71.7	59.4
Fraction	15.7	19.7	14.6	15.6	13.4	23.4	14.2	15.1	10.6	15.4
Decimal	20.7	0.1	18.6	17.6	40.4	4.1	17.1	9.0	6.7	15.8
Surd	1.1	7.5	13.2	5.7	3.2	0.8	7.7	15.3	9.1	8.1
Imaginary	0.0	0.0	4.2	0.0	0.0	0 0	1.9	2.4	1.8	1.4

In any practical problem that a boy will ever meet, the chances are against "coming out even." Harvard College recognized this fact in its old entrance examinations and used to insert a quadratic equation with roots coming out "uneven." It is a bad habit to fix—that of judging the correctness of an answer by its "coming out even"—but no change can be made using our present texts unless teachers demand it.

While considering the second characteristic of behavior (*i. e.*, mental set or determination) one must not ignore the fact that sets other than that of obtaining the correct answer are present. Of special significance are those more enduring sets that a pupil has with regard to his algebraic ability as a whole. The set or attitude—"I can (or cannot) do algebra," "I do (or do not) like mathematics" and the like—are of great importance in determining the character of work in algebra. It is likely that nearly all the attitudes can be traced back to some signal occasion of success or failure. The boy that is suggestible to the extent of believing that $(x^3)^4 = x^7$ by receiving a favorable nod from his teacher is suggestible to the extent of believing that he is a failure in algebra if he receives "E" on his first quarter's report. In so far as our marking system is reliable, this as a good thing—it discourages the unfit from proceeding with the

to them impossible. But since our marking system is so grossly unreliable as we know it to be, this undoubtedly discourages many otherwise hopeful cases.

Some may object to the statement that the set of a pupil on being presented with a problem is "to get the answer"—they would like to have it "to pass the term's work," or "to learn algebra," or some more distant end. It is true that these more ideal or distant ends may be the more controlling attitude or determination of the more intelligent or sagacious pupils, but for the great majority the set is immediate and direct, to be satisfied immediately and directly. If the more distant aims are to be built up, they must be built by training as must any other habit.

A third characteristic of behavior is the partial response to a situation. This is a very common phenomenon in algebra—it partly is what prescribes a certain response instead of the multitude of others that might be made to any example. Usually it is clearer to think of it as the prepotency of certain elements of a situation. To take a concrete example, consider the familiar:

$$5x^2 + 16x + 3 = (5x + 3)\ (x + 1)$$

Here the boy obtains $5x$ and x to multiply to give $5x^2$; and 3 and 1 to multiply to give 3; and having obtained an answer, is satisfied. We must not necessarily convict the boy of forgetfulness. He is in much the same situation as the young teacher before his first class confronted by a room full of wide awake youngsters, and also by a subject matter which must be organized, and by the details of class room management such as distributing papers, attending to ventilation, keeping records, etc. Would we convict our young teacher of forgetfulness if he failed to attend to the ventilation in his early training? Rather he has expended his maximum attention on details of teaching. So our boy in the example above may have struggled so in getting his result that the fact of the cross products escaped him altogether. Some other examples are

$$a^2 + 10a + 24 = (a + 8)\ (a + 2)$$

where the prepotency of getting $8a + 2a = 10a$ has over-

shadowed getting something to multiply to 24. A still more obvious example is

$$12y + 18y^4 = 6(2y + 3y^4)$$

where only the numerical factors are taken out and the common literal factors ignored. This law of partial response works especially with signs:

$$-4(3x - 4) = -12x - 16$$

can be explained by the ignoring of the effect of the minus sign of the first 4 on the minus sign of the second 4. This is the more apparent when we come to the failure to change signs when the numerator of a fraction is preceded by the minus sign in such an example as

$$\frac{4x - 2}{3} - \frac{x - 3}{4} = 0$$

$$16x - 8 - 3x - 9 = 0$$

Here the effect of the sign of the fraction on the -3 is ignored. Other illustrations of this partial activity where attention is given to an element of the situation and other elements ignored are: $(ab^2)^2 = ab^4$, where the effect of the square on the a is ignored; $\sqrt{\frac{3}{5}} = \frac{1}{5}\sqrt{3}$ where the concentration is on getting rid of the denominator under the radical with consequent ignoring of the numerator: $\sqrt{x^5y^8} = x^4y^8\sqrt{x}$, where the concentration is on the factoring process with consequent ignoring of the square root process:

$$\frac{Mt - g}{t} = M - g$$

where the presence of a t in both numerator and denominator is the prepotent stimulus, and the fact that t is not distributed throughout the numerator is ignored. Finally,

$$-12x = 25$$

$$x = \frac{25}{12}$$

is an exasperating error that comes under this same category—

the attention is concentrated on the division, and the sign is neglected.

A fourth characteristic of behavior is what Thorndike calls "response by analogy." In any new situation one responds as he would respond to the situation most nearly resembling it in his past experience. Errors in algebra illustrate this. Consider the following:

$$x^2 - 81 = -0$$
$$x = 9$$

In linear equations, the result comes out $x =$ something, and the single value of x is a satisfactory result; consequently in quadratic equations the natural first response is to be satisfied with a single value of x. This error is also seen when a solution for a single letter is considered sufficient in solving simultaneous equations. Indeed, nearly all cases of unfinished examples can be traced back to a previous process where what is now a step in our unfinished example was then the final result. Even the answer $-x = 12$ can be so explained. Usually the pupil has his solution on dividing by the coefficient of x, i. e. $6x = 72$, $x = 12$, and consequently when he has divided by the coefficient 6 in $-6x = 72$, he has the cue that he has finished the example.

$$x^2 - 81 = 0$$
$$(x + 9)\ (x - 9) = 0$$

is a case of response by analogy—the usual response to $x^2 - 81$, combined with an ignoring of the

$$n^2 - 7n = 12$$
$$n(n - 7) = 12$$

is another case of the same nature.

$$V = Lwh$$
$$w = V - Lh$$

is a wrong application of the rule—"to get anything from one side of an equation to the other, change the sign."

Under this same consideration come those errors which are made unless a thing is taught specifically.

$$(a^2 + 3)\ (a^2 - 3) = a^2 - 9$$

a^2 is written in the answer because all of the previous examples of product and difference of two numbers have involved only numbers of the first power.

$$(3 \cdot 2)^2$$
$$= 3 \cdot 4$$
$$= 12$$

comes because the previous response to 2^2 has been 4.

$$x^7 \cdot x = x^7$$

comes because the previous connections have all been—to add visible exponents.

This analysis of errors makes prominent the fact that learning algebra is an extremely complicated process. No wonder it often seems like juggling letters. When a definite response has been made to a combination of letters and symbols, we may expect this response to persist until a definite effort towards change is made in it. Oftentimes these errors show how well previous processes have been learned. They surely show us how we have failed in emphasizing those elements of a problem that require a new or different response.

A fifth characteristic of behavior is associative shifting, the learning process itself. Out of the number of various responses that a pupil may make when confronted with a given problem, that one is selected which gives its satisfaction in the form of teacher approval, or correspondence with the answer in the answer book, or verification by checking, or fitting in with other known algebraic and quantitative facts. And by drill and repetition, these correct responses become definitely established habits.

A sixth characteristic of behavior—perseveration—is extremely important to explain certain of the so-called "accidental" errors. By perseveration is meant "a tendency to repeat an act time after time, when once it has been aroused by some appropriate stimulus . . . Perseveration is most apt to

lead to recall of recent experiences, and is most apt to appear when attention is relaxed, and the mind allowed to run freely. Instances of perseveration are found in the running of a tune in the head, soon after it has been heard; or in the flashing of a scene before the mind's eye soon after it has been actually seen; or in the reminiscences of the day which are apt to come to mind as one is dropping off to sleep." This perseveration is apt to occur in algebra whenever the pupil is not working at top notch enthusiasm, or when he is disturbed in any way by other trains of thought, be they relating to outside activities or to what is going on in the classroom besides the problem at hand. Occasionally, however, it may result from a too great concentration upon a particular feature of a problem. Arithmetical errors like $24 \cdot 8 = 182$ are undoubtedly explainable in this way—$8 \times 4 = 32$, and the 2 perseverates to the exclusion of the 3 and is combined with the $2 \times 8 = 16$, thus: $16 + 2 = 18$. Errors in writing, such as

$$(2x - 3)^2 = 4y^2 - 12x + 9$$

may come from perseveration from some previous example.

$$\begin{aligned} &-4(3x - 4) \\ &= -12x - 16 \end{aligned}$$

may be a case of the perseveration of the minus sign as well as a case of partial activity.

$$\begin{aligned} y^2 + y &= 6 \\ y &= -3 \text{ or } -2 \end{aligned}$$

In cases where a boy does all of the work mentally, as in the above example, some element is quite apt to be perseverated, especially signs. In the above case, it may have been the minus sign before the 6 which he has to imagine when the 6 is transposed, or it may have been the minus sign of the 3 which he has just written. Such cases as

$$\sqrt{32} = 4\sqrt{4} \text{ and } \sqrt{50} = 5\sqrt{5}$$

rather than being errors in factoring, as Rugg and Clark suggest, point to a perseveration of the numerical value of the square root upon which he is concentrating. The boy does not write 16×2, but holds the confused picture in his mind, and in the confusion the 2 is lost.

I believe a seventh characteristic of behavior can be found which we may call *anticipation*. By this we mean the focussing of the attention on some element ahead of the writing in the process, which causes the reaction to occur before it should. This is the familiar error in speech when one says "prace rejudice" for "race prejudice," or "the stars have carted" for "the cars have started," or "fight the lires" for "light the fires." This comes usually at a time of special mental stress or mental eagerness—but as in perseveration, the attention is diverted from the mechanical reaction of speech in the present. It is hard to point out actual errors in algebra due to this cause—they are always "accidental" errors. Errors of reading, writing, and arithmetic may come under this heading. This phenomenon undoubtedly is a large factor in causing careless errors in examinations. For though the eagerness usually felt in an examination could not be called day-dreaming or mind-wandering, it might divert the attention from the specific process at hand.

The outcome of this psychological analysis of errors in algebra is that it enables us to improve our teaching in a way that epithets of "stupid," "careless," or "lazy" can never do. In the first place, it shows us the need for a psychological analysis of algebraic processes into the constituent connections or bonds involved. Especially those bonds that require a different response than in previous processes need analysis in order to receive emphasis in future teaching and drill. x^2x^7 without teaching would perhaps receive the answer x^{14}, because the customary thing to do when multiplying is to multiply the numbers on the scene. To add these exponents and obtain x^9 is an entirely new response in connection with numbers used in a multiplication process.

In the second place, our analysis of errors shows the need for drill or practice in various processes. Telling a boy what things to do does not form habits of doing them. The teacher ought not to expect that his explanation of a process is going to guarantee that the process is learned or a habit formed until the pupil has had a certain amount of drill on the process, not in fact until a test shows a certain proficiency in that process. Drill or practice is the remedy for attention to certain ele-

ments of an example to the exclusion of certain other elements. Not until certain processes become automatic can we expect the attention to be free to take in all the different steps needed in a solution. So the boy who omits the minus sign in solving $7x = -25$ needs more drill on short division so that the division will come automatically and attention can also take in the minus sign.

Whether we ought to attack a process like addition of fractions by learning many specific elements of the process, gradually combining and building up, or whether we should plunge the pupil at once into the final process, is still in doubt. It has been demonstrated that in learning poetry, nonsense syllables, etc., the whole method is superior to the part method. Poetry is a set of serial bonds, and usually there are no interferences from past connections to contend with. But algebra not only means the building up of a series of bonds, but bonds, some of which have facilitations from processes previously learned, and some of which have inhibitions from processes previously learned. There must be differential drill, with more emphasis on those connections that are hindered by past responses, and relatively less emphasis on those connections that are helped by past responses. The wise teacher will provide special drills of this propaedeutic nature.

Since much drill is necessary, and since only correct responses help build up correct habits, one should provide drill material that actually induces repetition of correct associations. I am referring to the custom of giving pupils a few difficult examples to do instead of many easy ones.

$$\left\{\sqrt[5]{\frac{a^{\frac{1}{2}}x^{-2}}{x^{\frac{1}{2}}a^{-2}}} \times \frac{a\sqrt{x}}{x^{-1}\sqrt{a}}\right\}^{-4}$$

Instead of giving 5 problems comparable to the needless one above, where the pupil is apt to stumble about in the maze of exponents and radicals, it is better first to make sure that he has acquired a certain degree of proficiency with easy processes such as

$$x^{-2} = \frac{1}{x^2} \quad \text{and} \quad \sqrt{x} = x^{\frac{1}{2}}, \text{ etc.}$$

In the third place, since accidental errors are largely due to mind wandering, our psychological analysis shows the need for a reorganization of our classroom procedure. We need to develop, as Rugg and Clark say, a "spirited atmosphere" in the classroom. This is possible by (1) timing all formal activities. Work should be done under pressure. But to complement this Rugg and Clark show that we need (2) rapid shifts in the type of mental process involved. Instead of dragging through the whole hour on the involved difficulties of one topic it is better to vary the work, giving part of the time to rapid review drill, and part to the taking up of some new topic.

In closing, let me say a word in regard to the criticism that will undoubtedly be raised with regard to this psychological analysis of errors: that no allowance has been made for the reasoning capacity of the pupil. A child who makes the mistake $-4(3x-4) = -12x-16$ is stupid or careless, they say, because the function of the parenthesis is to distribute operations to everything within it. And so all errors offend the teacher's sense of the meaning of algebraic symbols. But let us not forget that we who teach mathematics know the very things that our pupils do not know. We see these mistakes to be inconsistent, because we have learned the meanings and uses of the symbols and see their interrelations—they do not see them to be errors because they have not learned the meanings of the symbols and do not comprehend their significance. Meaning comes with experience, and things take a significance after we have become adjusted to them. We as teachers have been assuming that our pupils acquire these meanings and relationships and apply them in their processes and manipulations. Psychology teaches us that the opposite is really the truth: our pupils first learn to carry out the processes and manipulations, and thereby acquire the meaning and significance of what they do.